Table 1 Thermal load parameter for simply supported column

a/r	γ Finite element solution Four elements[a]	Eight elements[a]	Rayleigh-Ritz solution
0.0	1.0000	1.0000	1.0000
0.2	1.0100(3)	1.0100(3)	1.0100
0.4	1.0399(5)	1.0400(5)	1.0400
0.6	1.0898(5)	1.0900(5)	1.0900
0.8	1.1598(6)	1.1600(7)	1.1600
1.0	1.2496(7)	1.2500(8)	1.2500
λ_L	9.8746	9.8699	9.8696
$\bar{a}$	0.2496	0.2500	0.25

[a]Numbers in the parentheses indicate the number of iterations required to achieve an accuracy of 10^{-4} of the converged solution.

Table 2 Thermal load parameter for clamped and simply supported/clamped columns

a/r	γ Clamped Four elements[a]	Clamped Eight elements[a]	Simply supported/clamped Four elements[a]	Simply supported/clamped Eight elements[a]
0.0	1.0000	1.0000	1.0000	1.0000
0.2	1.0024(3)	1.0025(3)	1.0059(4)	1.0059(4)
0.4	1.0098(4)	1.0100(5)	1.0236(6)	1.0237(5)
0.6	1.0221(5)	1.0225(6)	1.0530(6)	1.0533(7)
0.8	1.0392(5)	1.0399(5)	1.0942(7)	1.0948(9)
1.0	1.0613(5)	1.0624(5)	1.1472(9)	1.1480(9)
λ_L	39.7753	39.4985	20.2322	20.1934
$\bar{a}$	0.0613	0.0624	0.1472	0.1480

[a]Numbers in the parentheses indicate the number of iterations required to achieve an accuracy of 10^{-4} of the converged solution.

$[=(\alpha TL^2/r^2)_{NL}$, α being the coefficient of thermal expansion, T the rise in temperature over the stress-free state, L the length of the column, and r the radius of gyration] is the eigenvalue giving the thermal load parameter in the nonlinear range.

As the nonlinear elastic stiffness matrix $[K]$ contains the unknown displacements and their derivatives (because of nonlinear strain-displacement relations considered), an iterative method is adopted to evaluate the matrix $[K]$. Starting with the iterative procedure, the nonlinear terms in $[K]$ are taken to be zero and the solution of Eq. (1) giving the linear thermal buckling load parameter λ_L $[=(\alpha TL^2/r^2)_L]$ is obtained. The linear eigenvector $\{\delta\}_L$ is then suitably scaled up by a scalar a which is the central lateral displacement of the column and is used to obtain the nonlinear elastic stiffness matrix $[K]$. Equation (1) is solved with the new matrix $[K]$ to give λ_{NL} corresponding to the central deflection a.

Using the above formulation and the iterative method, the linear thermal buckling parameter λ_L and the thermal load parameter λ_{NL} in the nonlinear range (for various values of a/r) are calculated for simply supported, clamped, and simply supported/clamped columns. Also, an empirical formula for γ $(=\lambda_{NL}/\lambda_L)$ in the form

$$\gamma = 1+\bar{a}(a/r)^2 \tag{2}$$

where $\bar{a}$ is a constant, is evaluated using the least-squares method from the values of γ obtained for various a/r values. Table 1 gives the values of λ_L and γ (for a/r ranging between 0.0 and 1.0 in increments of 0.2) for a simply supported column for four- and eight-element idealizations (of the full column). The $\bar{a}$ value of the empirical formula for γ is also given.

In order to establish the accuracy of the finite element formulation described above, a Rayleigh-Ritz analysis is carried out for the case of simply supported columns. The usual displacement distributions for the axial and transverse displacements, in terms of trigonometric functions, are assumed, which satisfy the geometric boundary conditions. Using these displacement distributions and minimizing the total potential energy, a system of nonlinear algebraic equations is obtained which yields the solution for λ_{NL}/λ_L as

$$\gamma = \frac{\lambda_{NL}}{\lambda_L} = 1+\frac{1}{4}\left(\frac{a}{r}\right)^2 \tag{3}$$

where $\lambda_L = \pi^2$.

In view of the assumed displacement distributions for simply supported columns, this solution is considered to be exact. A comparison of the finite element results (Table 1) for eight-element idealization with these results shows an excellent agreement and thus establishes the accuracy of the finite element scheme employed.

The results for clamped and simply supported clamped columns are presented in Table 2 for both four- and eight-element idealization (in full columns). The convergence of the results can be seen to be very good and as such an eight-element solution can be taken to be very accurate.

The difficulty in choosing appropriate displacement distributions for the Rayleigh-Ritz analysis can be overcome by this alternate simple finite element formulation. The results show that the effect of nonlinearity on the load ratios decreases from simply supported to simply supported/clamped and then to clamped columns.

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Vibration of Orthotropic Thick Circular Plates

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Introduction

LARGE amplitude vibration studies of circular plates have been carried out by Nowinski,[1] Yamaki,[2] Wah,[3] and several others.[4] While most of the investigations are concerned with isotropic plates, some deal with orthotropic circular plates as well. Classical nonlinear thin plate theory

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has been modified to study the nonlinear vibration behavior of rectilinearly orthotropic thin circular plates in Ref. 1. The results reported thus far do not, in many cases, incorporate the effects of transverse shear and rotatory inertia. Wu and Vinson[5] included these effects in the nonlinear vibration studies of composite rectangular plates using a Berger-type approximation. An improved version of this Berger-type theory recently has been suggested by Sathyamoorthy[6] for the dynamic analysis of rectilinearly orthotropic moderately thick circular plates. Numerical results indicate that the effects of transverse shear deformation and rotatory inertia are very important for moderately thick plates, particularly when the plate material is orthotropic, anisotropic, or composite.

This Note is concerned with the large amplitude flexural vibration of moderately thick orthotropic circular plates clamped along the boundary. Effects of transverse shear deformation and rotatory inertia have been incorporated into Berger's approach for orthotropic plates and a multiple-mode approach is used to investigate the nonlinear dynamic behavior. A solution for w is chosen in the polynomial form to satisfy the appropriate boundary conditions and the Berger-type nonlinear governing equations are integrated in conjunction with the immovable, nonlinear, in-plane boundary conditions. Numerical results applicable for moderately thick plates are presented for boron-epoxy and isotropic plates. Results corresponding to thin plates are also presented to illustrate clearly the effects of transverse shear and rotatory inertia. These effects, along with the effects of geometric nonlinearity, modal interaction, material properties, and plate parameters on large amplitude vibration behavior, are discussed. Similar results are obtained by means of a von Kármán-type theory that is much more complicated. Comparisons are made between these two theories to show the validity of Berger approximation for moderately thick plates.

Analysis

The governing equations applicable for a moderately thick rectilinearly orthotropic plate in Berger-type approximation can be written as[6]

$$u,_x^0 + \tfrac{1}{2} w,_x^2 + k(v,_y^0 + \tfrac{1}{2} w,_y^2) = \delta^2 h^2/12 = e \tag{1}$$

$$\begin{aligned} & a_1 I,_{xxxx} + a_2 I,_{xxyy} + a_3 I,_{yyyy} + a_4 I,_{tttt} \\ & + a_5 I,_{xxtt} + a_6 I,_{yytt} + a_7 I,_{xx} + a_8 I,_{yy} \\ & + a_9 I,_{tt} - I + a_{10}(w,_{xxtt} + w,_{yytt}) \\ & + a_{11} w,_{xxxx} + a_{12} w,_{xxyy} + a_{13} w,_{yyyy} + a_{14} w,_{xxyytt} \\ & + a_{15} w,_{xxxxtt} + a_{16} w,_{xxtttt} + a_{17} w,_{yyyytt} \\ & + a_{18} w,_{yytttt} + a_{19} w,_{xxxxxx} + a_{20} w,_{xxxxyy} \\ & + a_{21} w,_{yyyyxx} + a_{22} w,_{yyyyyy} = 0 \end{aligned} \tag{2}$$

where

$$I = q(x,y) - \rho h w,_{tt} + C_1 e(w,_{xx} + k w,_{yy})$$

$$C_1 = E_x h/\mu, \quad \mu = 1 - \upsilon_{xy}\upsilon_{yx}, \quad k = (E_y/E_x)^{1/2} \tag{3}$$

The coefficients a_i in Eq. (2) are defined in Ref. 6. These coefficients contain tracing constants T_s and R_i which should be taken as unity for moderately thick plates. $T_s = R_i = 0$ for classical nonlinear thin plates, and it can be shown that Eqs. (1) and (2) will readily reduce to the Berger-type equations applicable to thin plates. E_x, E_y, υ_{xy}, υ_{yx} are the elastic constants of the orthotropic plate material; h the thickness of plate; ρ the mass density; and $q(x,y)$ the external load on the plate. Similar equations corresponding to the von Kármán-type plate theory are:

$$f,_{xxxx} + k^2 f,_{yyyy} + m^2 f,_{xxyy} = E_y(w,_{xy}^2 - w,_{xx} w,_{yy}) \tag{4}$$

The second equation is obtained by replacing I with $\bar{I}$ in Eq. (2) where

$$\bar{I} = q(x,y) - \rho h w,_{tt} + h(f,_{yy} w,_{xx} + f,_{xx} w,_{yy} - 2f,_{xy} w,_{xy})$$

and

$$q^2 = \upsilon_{yx}, \quad p^2 = \mu G_{xy}/E_x, \quad m^2 = (k^2 - q^4 - 2p^2q^2)/p^2 \tag{5}$$

In Eq. (5) f is the stress function and G_{xy} the shear modulus. In what follows, governing equations corresponding to these two different theories are solved using a multiple-mode approach.

A mode shape for w is assumed in the following multiple-mode polynomial form.

$$\begin{aligned} w(x,y,\tau) = {} & (h/a^4)(x^2+y^2-a^2)^2 [A_1(\tau) \\ & + (A_2(\tau)/a^2)(x^2+y^2-a^2)] \end{aligned} \tag{6}$$

where $A_1(\tau)$ and $A_2(\tau)$ are unknown functions of nondimensional time τ such that $\tau^2 = t^2 E_x/\rho a^2$. Equation (6) is substituted in Eq. (1) and integrated with the aid of immovable conditions along the boundary, i.e., $u^0 = v^0 = 0$. This procedure results in an expression for e which is further used in Eq. (2) along with Eq. (6), and Galerkin's method is applied to obtain two nonlinear time-differential equations in $A_1(\tau)$ and $A_2(\tau)$. These equations are the modal equations corresponding to the Berger-type theory. Similarly, Eq. (6) is substituted in Eq. (4) and an exact solution for f is sought in a polynomial form consisting of 27 coefficients that are determined using Eq. (4) and the immovable boundary conditions ($u^0 = v^0 = 0$) expressed in terms of f. Such a procedure will completely define the stress function f. Using this f and the w given by Eq. (6) in Eq. (2) (where I is replaced by $\bar{I}$), and following Galerkin's method, two more nonlinear time-differential equations can be obtained. The two sets of modal equations take the same form as given below but the coefficients are different for the two theories considered herein.

$$\begin{aligned} & (b_1,c_1)(A_1^3),_{\tau\tau\tau\tau} + (b_2,c_2)(A_1^2 A_2),_{\tau\tau\tau\tau} \\ & + (b_3,c_3)(A_1 A_2^2),_{\tau\tau\tau\tau} + (b_4,c_4)(A_2^3),_{\tau\tau\tau\tau} \\ & + (b_5,c_5)(A_1),_{\tau\tau\tau\tau\tau\tau} + (b_6,c_6)(A_2),_{\tau\tau\tau\tau\tau\tau} \\ & + (b_7,c_7)(A_1^3),_{\tau\tau} + (b_8,c_8)(A_1^2 A_2),_{\tau\tau} + (b_9,c_9)(A_1 A_2^2),_{\tau\tau} \\ & + (b_{10},c_{10})(A_2^3),_{\tau\tau} + (b_{11},c_{11})(A_1),_{\tau\tau\tau\tau} \\ & + (b_{12},c_{12})(A_2),_{\tau\tau\tau\tau} \\ & + (b_{13},c_{13})A_1^3 + (b_{14},c_{14})A_1^2 A_2 \\ & + (b_{15},c_{15})A_1 A_2^2 + (b_{16},c_{16})A_2^3 + (b_{17},c_{17})(A_1),_{\tau\tau} \\ & + (b_{18},c_{18})(A_2),_{\tau\tau} + (b_{19},c_{19})A_1 + (b_{20},c_{20})A_2 = q_0^* \end{aligned} \tag{7,8}$$

Where q_0^* is the nondimensional load $q_0\beta^4/E_x$ where $\beta = a/h$. Equations (7) and (8) are applicable for moderately thick orthotropic circular plates where $T_s = R_i = 1$. If the effects of transverse shear and rotatory inertia are ignored, i.e., $T_s = R_i = 0$, then the coefficients b_1-b_{12} and c_1-c_{12} become zero and the resulting modal equations are applicable for orthotropic thin circular plates. In the case of static nonlinear problems, a set of nonlinear algebraic equations can be

Table 1 Values of period ratios $(T/T_0)10^4$ for isotropic circular plates by stress function approach

w_{max}	Thick plates ($T_s = R_i = 1$) $\beta = 20$		$\beta = 30$		Thin plates ($T_s = R_i = 0$)		
h	SMS[a]	MMS[a]	SMS	MMS	SMS	MMS	Ref. 2
0.0	10,042	10,117	10,023	10,053	10,000	10,000	10,000
0.5	9,621	9,628	9,600	9,571	9,586	9,530	9,584
1.0	8,629	7,834	8,621	7,856	8,608	7,862	8,586
1.5	7,501	6,090	7,500	6,082	7,494	6,074	7,460
2.0	6,483	5,483	6,484	5,442	6,484	5,863	6,510

[a] SMS – single mode solution, MMS – multiple mode solution.

Table 2 Values of period ratios $(T/T_0)10^4$ for isotropic circular plates by Berger approach

w_{max}	Thick plates ($T_s = R_i = 1$) $\beta = 20$		$\beta = 30$		Thin plates ($T_s = R_i = 0$)		
h	SMS	MMS	SMS	MMS	SMS	MMS	Ref. 6
0.0	10,042	10,117	10,023	10,053	10,000	10,000	10,000
0.5	9,597	9,732	9,576	9,676	9,563	9,630	9,562
1.0	8,560	8,299	8,554	8,300	8,541	8,294	8,521
1.5	7,402	6,176	7,401	6,178	7,397	6,177	7,369
2.0	6,369	5,297	6,372	5,290	6,373	5,284	6,420

Table 3 Values of period ratios $(T/T_0)10^4$ for boron-epoxy circular plates by stress function approach

w_{max}	Thick plates ($T_s = R_i = 1$) $\beta = 20$		$\beta = 30$		Thin plates ($T_s = R_i = 0$)	
h	SMS	MMS	SMS	MMS	SMS	MMS
0.0	10,178	10,504	10,078	10,230	10,000	10,000
0.5	9,719	9,947	9,648	9,750	9,590	9,577
1.0	8,669	7,782	8,643	7,879	8,617	7,937
1.5	7,491	6,141	7,505	6,116	7,508	6,090
2.0	6,443	5,481	6,467	5,470	6,499	5,468

Table 4 Values of period ratios $(T/T_0)10^4$ for boron-epoxy circular plates by Berger approach

w_{max}	Thick plates ($T_s = R_i = 1$) $\beta = 20$		$\beta = 30$		Thin plates ($T_s = R_i = 0$)	
h	SMS	MMS	SMS	MMS	SMS	MMS
0.0	10,178	10,504	10,078	10,230	10,000	10,000
0.5	9,691	10,035	9,621	9,819	9,565	9,634
1.0	8,593	8,273	8,568	8,316	8,545	8,301
1.5	7,382	6,142	7,472	6,178	7,402	6,184
2.0	6,321	5,291	6,355	5,297	6,379	5,288

readily obtained for each case treating A_1 and A_2 as time-independent. Solutions to these algebraic equations will give the nonlinear load-deflection relations for both thick and thin plates. The coupled nonlinear modal equations (7) and (8) are integrated numerically using an IMSL subroutine DVERK which is based on the numerical Runge-Kutta procedure. Results are reported for moderately thick as well as thin plates considering the two different approaches, i.e., the stress function approach and the Berger approach.

Results

Variations of period ratios with amplitude are tabulated in Tables 1-4 for large amplitude vibration of isotropic and boron-epoxy circular plates. Effects of transverse shear and rotatory inertia are included in the nonlinear period T, whereas these effects are ignored when calculating linear periods T_0. Values of material constants E_x/E_y, ν_{xy}, G_{xy}/E_x, G_{xz}/E_x, and G_{yz}/E_x for isotropic plates are 1, 0.3, 0.3846, 0.3846, and 0.3846, respectively. Similar values for boron-epoxy plates are 10, 0.22, 0.033, 0.033, and 0.0165. A comparison of numerical values indicates that the effects of transverse shear deformation and rotatory inertia on the large-amplitude vibration of circular plates are shown by an increase in the period ratio although the increase is less at moderately large amplitudes. For the cases considered here, the period-amplitude variation follows a hardening type pattern. Multiple-mode solutions show a pronounced nonlinearity compared to single-mode solutions and this behavior is attributed to modal coupling. This type of behavior is common to both stress function and Berger approaches. Period ratios predicted by the Berger approach are generally lower than those predicted by the stress function approach, which is in agreement with previous observations.[4,6] It is possible to solve one of the nonlinear governing equations exactly in the stress function approach, whereas the Berger approach requires much less effort in the solution procedure.

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Structural Optimization by Mathematical Programming Methods

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Introduction

A COMPREHENSIVE study of mathematical programming methods as they apply to structural (mechanical) system design optimization has been completed recently under an NSF grant.[1] The major objective of the study was to develop a unified viewpoint of the modern mathematical programming (MP) methods as they apply to such engineering design problems. Each method is analyzed to determine its strengths and weaknesses. The methods analyzed are taken from the existing MP literature and, therefore, are not new. What is new is the analytical and numerical comparison of the methods, unified viewpoint, and development of a basis for studying methods as they apply to the engineering design environment. No "best method" is recommended since the meaning of "best" depends on an individual's viewpoint.

Very few comprehensive studies on comparison of methods for optimal design have been conducted. Some studies have been conducted in the MP literature.[1] However, those studies have the following limitations: 1) only small-scale problems are considered, 2) analytical aspects are not considered, 3) functions in the problem depend explicitly on design variables as opposed to implicit functions in structural design problems, and 4) problems with multiple local minima are not considered. The present study is conducted to overcome these limitations with a particular regard to structural design problems. The purpose of this Note is to present the findings of this study.

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Methods Considered

Only gradient-based methods are studied because quantities such as stress and displacement which enter into the structural optimization problem are usually continuously differentiable functions of design variables. Methods such as dymamic programming, geometric programming, physically motivated optimality criteria, and fully stressed design are not included in the study, since they are not suitable for general design applications. Also, methods requiring substantial interaction by the designer during the iterative process are not included in the study. While these methods are especially effective in certain situations, they are not general-purpose enough to be combined with analysis codes to form design systems.

Mathematical programming methods are based on solving the Kuhn-Tucker optimality conditions. They are classified into two categories: primal and transformation methods. The primal methods studied in this work are: 1) recursive quadratic programming (RQP) methods; 2) gradient projection methods; 3) reduced gradient methods; 4) method of Bard and Greenstadt; 5) feasible directions methods; 6) optimality criteria methods; 7) sequential linear programming (LP) methods; 8) projection methods; and 9) methods which solve nonlinear reduced problems. Transformation methods are conceptually different from primal methods. These methods transform the original constrained problem into a sequence of unconstrained problems. Within this category, the methods considered in the study are: 1) sequential unconstrained minimization techniques (SUMT) (penalty and barrier functions), and 2) multiplier (or augmented Lagrangian) methods.

Basis of Comparison

A basis for comparison of methods has been developed by considering features of the design problem and global convergence aspects. The basis is used to meet objectives of the study and consists of the following considerations.

1) Special structure possessed by a method. An essential difference between the structural optimization problem and the mathematical programming problem is that the functions in the former are implicitly dependent on design variables. The question therefore is whether or not there are methods whose structure is especially suited for handling such implicit functions.

2) Geometrical aspects. Almost all methods update the current design using an iterative process consisting of direction finding and step size determination problems. The geometrical significance of the direction vector for each method should be studied with a view toward bringing out similarities and differences between the methods.

3) Global convergence. The global convergence is a very desirable property for a method to possess. A method is said to be globally convergent if it converges to a solution from any starting design. Global convergence is an indication of reliability of a method. We need to examine whether the method possesses global convergence, and, if it does, what is its computational cost. Specifically, whether or not global convergence is achieved using an "active-set" strategy.

4) Active-Set strategy. In general, the direction vector at the current design point is determined using cost and constraint functions and their gradients. By an active-set strategy it is meant that, to determine the direction vector, only a subset of all the constraints is used at any iteration. Usually, the subset consists of only those constraints that are nearly satisfied or violated. The importance of using an active set is the savings in computation when the gradients of only a few (implicit) constraint functions have to be evaluated at each iteration.